Vagueness at all orders

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Outline of the talk

Joint work with Paul Égré

"Vagueness, Uncertainty and Degrees of Clarity"
Synthese, 2009.

Non-transitivity and Vagueness

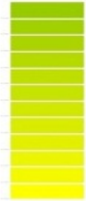
Intransitivity of indiscriminability
It may be the case that a sample A is indiscriminable in colour from a sample B, B is indiscriminable from C, and still A and C are not indiscriminable.

Vagueness may arise whenever there is non-transitivity
If A is categorized as ‘red’ and C as ‘not-red’, B seems destined to be a borderline case.

This holds of *direct discriminability*, one may put into question whether it applies to *indirect discriminability* as well.

Non-transitivity at work

- *Induction premiss in sorites paradox*
  $Px \wedge xRx' \rightarrow Px'$
- *Categorization effects in forced march*
  I am more likely to categorize
  some not so X thing as being X if,
  starting from a clearly X thing,
  I have been gradually led to do so.



Margins for errors (Williamson)

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Williamson makes a further move and suggests to give a semantics for $\Box$ by turning the *if* into an *iff*.

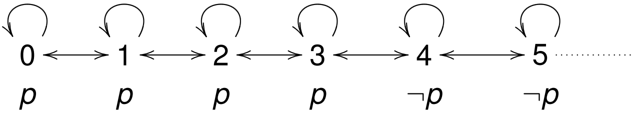
Margin for error semantics

A *fixed-margin* model is a 4-uplet $\langle W, d, \alpha, V\rangle$, where d is a metric over W, and α a real valued margin for error parameter.

$w \models \Box\phi$ iff for every w' such that $d(w, w') \leq \alpha$, $w' \models \phi$.

At a given world, the subject knows / it is clear that some proposition ϕ holds iff ϕ holds at every world included within the margin α from w.

A discrimination task

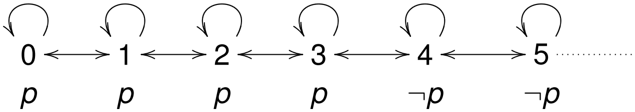


A series of pens, such that all and only pens that are less than 4 cm fit in a certain box.

A subject sees the pens and the box at a certain distance and is asked which pens will fit in the box.

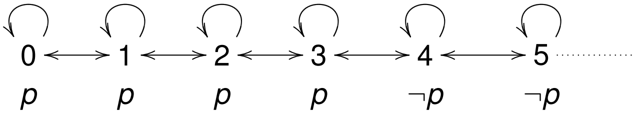
The subject can perceptually discriminate only between pens whose size differs by more than 1 cm. non-adjacent pens.

Inexact Knowledge



- $3 \models p$
- $3 \not\models \Box p$
- $3 \models \neg\Box p \wedge \neg\Box\neg p$
- $2 \models \Box p$

Vagueness spreads out



If $\Box$ is interpreted as 'the subject knows that' :
Introspection principles are not valid.
$2 \vDash \Box p$ but $2 \nvDash \Box\Box p$

If $\Box$ is interpreted as 'it is clear that' :
we get higher-order vagueness.
$2 \nvDash \neg\Box\Box p$ et $2 \nvDash \Box\neg\Box p$

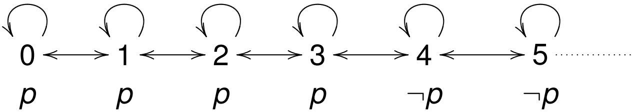
Resisting the Spread of Vagueness

- Iterations of Knowledge / Definiteness as ‘a process of gradual erosion’

vs

- KK as a rationality principle,
- Limited higher-order vagueness.

The Hidden Assumption



- ‘Williamsonian’ view

in order to determine whether $2 \models \Box\Box p$, alternatives to 3 which are *not* alternatives to 2 come into play.

- Supervenience view

in order to determine whether $2 \models (\Box\Diamond)p$, only alternatives to 2 should come into play.

Centered Semantics

Truth is first defined for couple of worlds :

(i) $\mathcal{M}, (w, w') \vDash_{\mathrm{CS}} p$ iff $w' \in V(p)$

(ii) $\mathcal{M}, (w, w') \vDash_{\mathrm{CS}} \neg\phi$ iff $\mathcal{M}, (w, w') \nvDash_{\mathrm{CS}} \phi$

(iii) $\mathcal{M}, (w, w') \vDash_{\mathrm{CS}} (\phi \wedge \psi)$
iff $\mathcal{M}, (w, w') \vDash_{\mathrm{CS}} \phi$ and $\mathcal{M}, (w, w') \vDash_{\mathrm{CS}} \psi$

(iv) $\mathcal{M}, (w, w') \vDash_{\mathrm{CS}} \Box\phi$
iff for all w'' such that wRw'', $\mathcal{M}, (w, w'') \vDash_{\mathrm{CS}} \phi$

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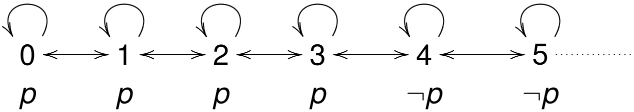
Then we set $\mathcal{M}, w \vDash_{CS} \phi$ iff $\mathcal{M}, (w, w) \vDash \phi$

Inexact Knowledge



- $(3,3) \not\models_{CS} \Box p$
  because $(3,4) \not\models_{CS} p$
- $(2,2) \models_{CS} \Box p$
  because $(2,3) \models_{CS} p$ and so do $(2,2)$ and $(2,1)$
- $(2,2) \models_{CS} \Box\Box p$ because $(2,3) \models_{CS} \Box p$ and so do $(2,2)$ and $(2,1)$

Axioms for CS

Fact

CS validity is axiomatized by K45

Fact

CS validity over reflexive frames is axiomatized by S5

Knowledge vs Clarity

Iteration patterns might differ for $\Box$ as 'The subject knows that' and $\Box$ as 'It is clear that'.

- accept KK
- reject higher-order vagueness *beyond some level*

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Generalizing *CS* yields a family of resource-sensitive logics, $TS(n)$ for *n Token Semantics*, where n bounds the number of allowed context shifts. In $TS(n)$, iterations of $\Box$ or $\Diamond$ beyond n are trivialized.

- *CS* is $TS(1)$,
- *ML* is $TS(\omega)$,
- A logic of clarity might be provided by $TS(n)$ for $1 < n < \omega$

Halpern's solution

We have changed the semantics. An alternative would be to change the models.

Halpern (2004) : Non-transitivity is the product of two more fundamental transitive 'similarity' relations.

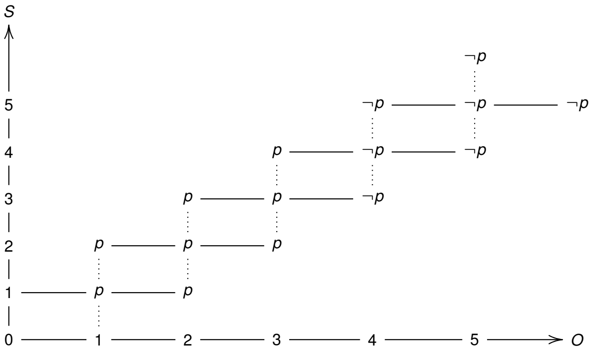
- a bidimensional modeling : external states and inner states,
- a relation $\sim_s$ describes the agent's uncertainty about the target value, given her subjective state.
- a relation $\sim_o$ which describes the fluctuation of the agent's estimate around the target value.

A Halpern model is a structure $\langle W, \sim_s, \sim_o, V\rangle$, with $W \subseteq O \times S$ where S is a set of subjective states and O a set of objective states.

and

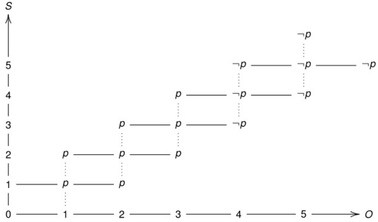
- $\sim_s$ is an equivalence relation btw pairs with the same subjective index,
  and interprets an operator D 'it is definitely the case that'.
- $\sim_o$ is an equivalence relation btw pairs with the same objective index,
  and interprets an operator R 'the agent reports that'.

Back to the discrimination task



- $(3,3) \vdash \neg Dp$
- $(2,2) \models Dp$ and $(2,2) \models DDp$,
- $(2,2) \models \neg DRDRp$.

Halpern semantics and Centered Semantics



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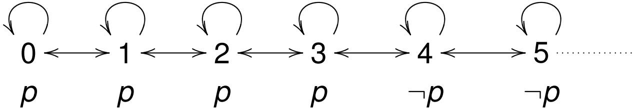


by dividing the structure into spheres of radius 1.

Layering

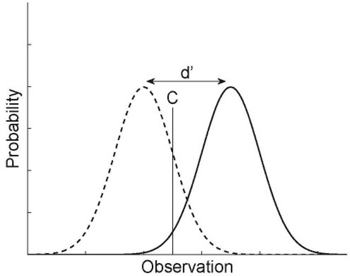
Given a Kripke model $M = \langle W, R, V\rangle$, let $L(M) = \langle W', R', V'\rangle$ be the *layering* of M defined by :

- $W' = \{(w, w') \in W \times W \;/\; w'Rw \vee w' = w\}$
- $(w, w')R'(u, u')$ iff $w' = u'$ and $w'Ru$; and finally, $(w, w') \in V'(p)$ iff $w \in V(p)$

Fact

$M, w \vDash_{\mathrm{CS}} \phi \iff L(M), (w, w) \models \phi$ *(for* $\Box = D$*).*

Signal Detection Theory



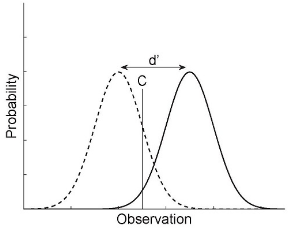
Signal Detection Theory
A statistical model, which is commonly used
in psychophysics to account for sensory processes,
and more generally to account for decisions in uncertainty.

A typical SDT plot

- dashed curve represents $P(x|n)$.
- continuous curve represents $P(x|s)$.

There is uncertainty when the probability distributions overlap. The smaller d', the more uncertainty there is.

C represents some decision criterion.
eg : categorize as s when $\frac{P(x|s)}{P(x|n)} > 1$.



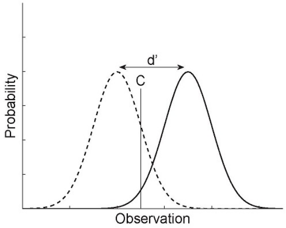
Prior and posterior probabilities

Prior probabilities of the form $P(x|A)$ constitute a quantitative version of $\sim_o$.

Posterior probabilities of the form $P(A|x)$ constitute a quantitative version of $\sim_s$.

Posterior probabilities can be computed from prior probabilities and probabilities for the objective states.

$$P(s|x) = \frac{P(s)\times P(x|s)}{P(s)\times P(x|s)+P(n)\times P(x|n)}$$



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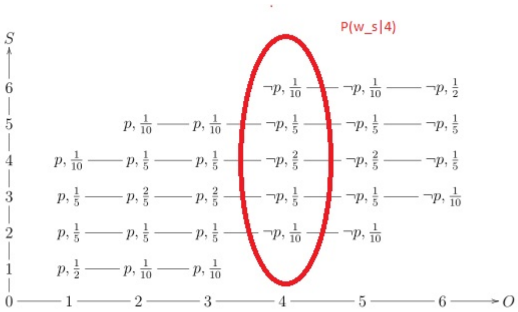
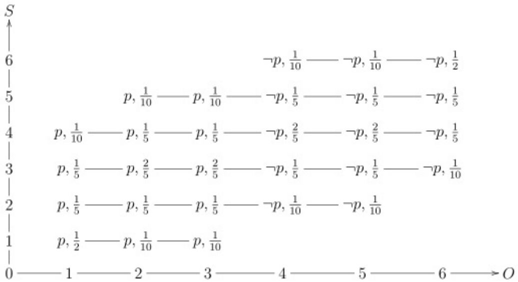


where the number in cell (x, y) is $P(x|y)$.

Deriving conditional probabilities for objective states



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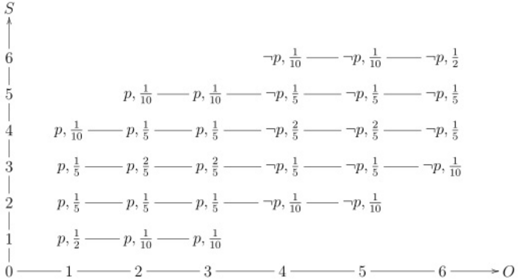

Posterior probabilities :

$$P(w_o|w_s) = \frac{P(w_s|w_o)}{\sum_y P(w_s|y)}$$

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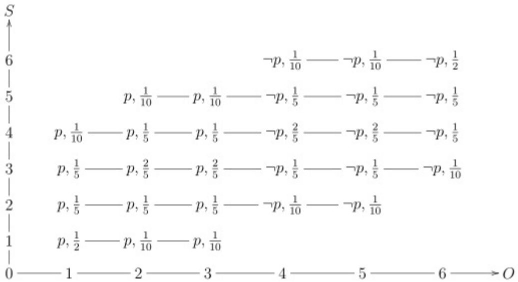

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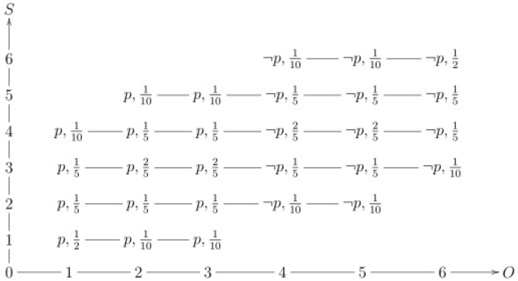

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$\longrightarrow$ for every $w \in S$, we get
a probability measure Π_w on $\{(y, w_s) \mid y \in O\}$,

From quantities to qualities

How can we define $(w, w') \models \Box\phi$ from $\Pi_{w'}$?

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  Something can be clear, or very clear, or extremely clear...

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(well-known) problem :
If we want $\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$
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Idea (Skyrms, Leitgeb, etc.) :
Require that $\Pi_{w'}(\phi)$ be robustly high.

Skyrm's solution

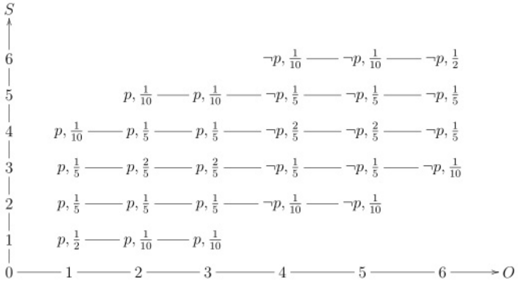
Definition (resistance)

Let the resistance $R(\phi)$ of ϕ be $Min(Pr(\phi|\psi))$ where ψ is compatible with ψ.

Theorem (Skyrms, 1980)

The set of formulas whose resistance is greater than α for $\alpha > 0.5$ is coherent and closed under conjunction.

Back to the discrimination task



Take $\alpha = 0,5$,

- $(2,3) \models \Box p$
- $(4,4) \not\models \Box p$

Iterations with probabilities

In order to say whether $(w, w') \models \Box\Box\phi$
one need to determine whether $(x, w') \models \Box\phi$
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as before, they are several options

- similar to Centered Semantics : supervenience
- similar to Williamson's : context-shift
- similar to Token Semantics : limited context-shift

Iterations with probabilities (cont.)

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  Whether $(x, w') \models \Box\phi$
  will depend on P_y for the y s.t. $P(y|x) \neq 0$
  but depend how ?
  - require $\Box\phi$ wrt all such P_y
  - require $\Box\phi$ with a (stably) high probability
- context-shift up to some point

Conclusion

- Indiscriminability may be at the heart of vagueness, but one cannot jump from facts about indiscriminability to semantic facts about knowledge / clarity,
- non-transitivity need not imply failure of KK / higher-order vagueness, and non-transitivity may be factored as a product of transitive relations,
- probabilistic models in psychophysics provide a natural background for a quantitative approach to the analysis of (perceptual) belief / clarity.