Contextual Models of Vagueness and Vague Quantifiers

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Dec 8, 2010 / Vague Quantities and Quantifiers

Linguistic models of vagueness

Kinds of approaches

Scale-based approaches

- Focus on gradable adjectives like *tall*, *wide*, *red*,...
- Individuals can posses a property to a certain measurable degree
- Scales are associated with the dimension referred to by an adjective
- Currently very popular in linguistics
- NB: No degrees of truth - not fuzzy logic

Delineation approaches

- Analyze gradable adjectives as simple predicates...
- ... whose extensions are crucially context dependent
- Popular in philosophy of vagueness

Linguistic models of vagueness

Roadmap

1. Pick two concrete scale- and delineation-based approaches
   - Ch. Barker (2002), The Dynamics of Vagueness and
   - Kyburg/Morreau (2000), Fitting words: Vague language in context
2. Point out how vague quantities and quantifiers can be incorporated
3. Show how these approaches relate to each other
   - I.e. In which situations they make the same predictions
4. Illustrate aspects where one approach is superior to the other
   - E.g. Predicate modifiers like *very*, *clearly*, ...
   - Accomodation

Scale based approaches

Ch. Barker (2002), The Dynamics of Vagueness

- Vague adjectives allow for measurement in degrees
- Degrees are measured on a corresponding scale

Main ingredients

- A context C is identified with a set of possible worlds
- A possible world $c \in C$ corresponds to a complete precisification of (relevant) adjectives
- Initially C holds all ways of making all adjectives precise
- During a conversation more and more possible worlds in C are filtered out
- Both descriptive and meta-linguistic meaning of utterances

Scale based approaches

Ch. Barker (2002), The Dynamics of Vagueness

More formally:

C	the context in which a sentence is evaluated
$w \in C$	a possible world
D_C	the relevant universe of discourse
$\delta(w):$	$Adj \rightarrow Deg$ delineation function, maps adjectives to threshold values

Additionally one constant relation for each vague adjective, e.g.

$\mathbf{tall}(w)(\mathbf{a})$	denoting the degree to which the individual **a** is *tall* in the world *w*.

For a sentence ϕ, its meaning $[\![\phi]\!]$ is a filter on contexts:

$[\![\phi]\!](C) \subseteq C:$ all possible worlds in C in which ϕ is true.

Scale based approaches

Ch. Barker (2002), The Dynamics of Vagueness

An example context

Consider the three predicates $[\![tall]\!]$, $[\![clever]\!]$, and $[\![heavy]\!]$ and the context $C = \{w_1, w_2, w_3, w_4, w_5\}$ with **f** refering to *Feynman*

w	$\delta(w)([\![tall]\!])$	$\mathbf{tall}(w)(\mathbf{f})$	$\delta(w)([\![clever]\!])$	$\mathbf{clever}(w)(\mathbf{f})$	$\delta(w)([\![heavy]\!])$	$\mathbf{heavy}(w)(\mathbf{f})$
w_1	170	175	100	105	80	75
w_2	160	170	120	125	75	70
w_3	170	180	100	95	90	100
w_4	180	175	105	100	85	75
w_5	170	165	110	115	70	65

In this situation, *Feynman* is *heavy* in w_3, *clever* in w_1, w_2, w_5, and *tall* in w_1, w_2, w_3.

- "Real-world contexts" are larger, but finite

Scale based approaches

Ch. Barker (2002), The Dynamics of Vagueness

Meaning of *tall*

$[\![tall]\!] =_{\text{DEF}} \lambda x \lambda C.\{w \in C : \delta(w)(^{\uparrow}[\![tall]\!]) \leq \mathbf{tall}(w)(x)\}$

Usage of $[\![tall]\!]$

$[\![\textit{Feynman is tall}]\!] = [\![tall]\!](\mathbf{f}) = \lambda C.\{w \in C : \delta(w)(^{\uparrow}[\![tall]\!]) \leq \mathbf{tall}(w)(\mathbf{f})\}$ with **f** referring to the individual *Feynman*.

Comparative as predicate modifier

$[\![\textit{-er}]\!] =_{\text{DEF}} \lambda \alpha \lambda x \lambda y \lambda C.\{w \in C : \mathbf{a}(w)(x) \leq \mathbf{a}(w)(y)\}$

- Truth of the sentence ϕ at the world w in the context C is identified with $w \in [\![\phi]\!](C)$.

Scale based approaches

Ch. Barker (2002), The Dynamics of Vagueness

Other predicate modifiers

- $[\![very]\!] =_{\text{DEF}} \lambda\alpha\lambda x\lambda C.\{w \in \alpha(x)(C) :$
  $\delta(w)([\![very]\!]) + \delta(w)(\alpha) \leq \mathbf{a}(w)(x)\}$
- $[\![definitely]\!] =_{\text{DEF}} \lambda\alpha\lambda x\lambda C.\{w \in \alpha(x)(C) :$
  $\forall d(w[d/\alpha] \in C \rightarrow w[d/\alpha] \in \alpha(x)(C))\}$
- $[\![clearly]\!] =_{\text{DEF}} \lambda\alpha\lambda x\lambda C.\{w \in \alpha(x)(C) :$
  $(\delta(w)([\![clearly]\!]) + \max_\alpha \leq \max_C)\}$

with

- $w[d/\alpha]$ denoting a world that is exactly like w, except for setting $\delta(w)(\alpha) = d$,
- $\max_\alpha = \max\{d : \alpha(x)(\{w[d/\alpha]\}) \neq \emptyset\}$ and
- $\max_C = \max\{d : w[d/\alpha] \in C\}$.

Scale based approaches

Concerning vague quantities and quantifiers

- Barker does not consider vague quantitfiers
- Extend Barker's framework by:
  - Predicates of the type $P_C : C \rightarrow D_C \rightarrow \{t, f\}$

Example predicates

- $[\![tall]\!] = \lambda w.\lambda x.\{\delta(w)([\![tall]\!]) \leq \textbf{tall}(w)(x)\}$
- $[\![heavy]\!] = \lambda w.\lambda x.\{\delta(w)([\![heavy]\!]) \leq \textbf{heavy}(w)(x)\}$

  - "Vague quantifiers" of the type $VQ_C : P_C \rightarrow P_C \rightarrow C \rightarrow C$

Example sentence

- $[\![\textit{Many tall people are heavy}]\!] = \lambda C.[\![many]\!]([\![tall]\!])([\![heavy]\!])(C)$

Scale based approaches

Concerning vague quantities and quantifiers

Vague quantifiers

- $[\![many]\!] =_{\text{DEF}} \lambda P_1.\lambda P_2.\lambda C.\{w \in C :$
  $|\{x \in D_C : P_1(w)(x) \wedge P_1(w)(x)\}|/|\{x \in D_C : P_2(w)(x)\}|$
  $\geq \delta(w)([\![many]\!])\}$

- Analysis: "*Many A are B*" means that the ratio of $A \cap B$ and B is higher than some local standard of "*many*ness".
- Analogous to Barker's analysis of *very*
- Similar for $[\![few]\!]$, $[\![most]\!]$:
  at all possible worlds w the following should hold:
  $\delta(w)([\![few]\!]) < \delta(w)([\![many]\!]) < \delta(w)([\![most]\!])$

Scale based approaches

Concerning vague quantities and quantifiers

- Introduce vague quantities analogously:

Vague quantities

- $[\![\textit{roughly}]\!] : \mathbb{N} \to P_C \to C \to C$
  $[\![\textit{roughly}]\!] =_{\text{DEF}} \lambda n.\lambda P.\lambda(C).\{w \in C :$
  $\quad abs(|x \in D_C : P(w)(x)| - n) \leq \delta(w)([\![\textit{roughly}]\!])\}$

- Also possible to model the round number effect:
- Define $[\![n]\!]$ analogously using the vague standards $\delta(w)([\![10]\!]) \leq \delta(w)([\![50]\!]) \leq \delta(w)([\![100]\!])$,... for n divisible by 10,50,100,... respectively
- Also other possibilities: Model shifting of vague standards

Scale based approaches

Concerning vague quantities and quantifiers

- Alternative characterization of round numbers:

Round numbers

- $[\![n]\!] =_{\text{DEF}} \lambda P.\lambda(C).\{w \in C :$
  $abs(|x \in D_C : P(w)(x)| - n) \leq \delta(w)([\![NUM]\!]) \wedge$
  $\delta(w)([\![NUM]\!]) \leq \delta(w)([\![n]\!])\}$

- Granularity of the scale is modeled by $\delta(w)([\![n]\!])$
- $\delta(w)([\![10]\!]) = \delta(w)([\![20]\!]) = \ldots$
  $\leq \delta(w)([\![100]\!]) = \delta(w)([\![200]\!]) = \ldots$
  $\ldots$
- Vague standard of granularity can be sharpened during a context

Delineation approaches

Kyburg/Morreau (2000), Fitting words: Vague language in context

- Vague adjectives are analyzed as simple predicates...
- ...whose extensions are crucially context dependent

Main ingredients

- A context p is identified with a partial interpretation:
- Predicates may be true, false, or undefined
- Explicitly model the context space $\mathscr{P}$ as the space of all admissible contexts
- Use supervaluation for characterizing truth
- Perform belief revision for accomodation

Delineation approaches

Kyburg/Morreau (2000), Fitting words: Vague language in context

More formally:

A model is defined as a structure $< \mathcal{U}, \mathcal{R}, \mathscr{P}, \leq, \mathcal{I}, \mathscr{M} >$ where:

$\mathcal{U}$	the universe of discourse
$\mathcal{R}$	the set of relevant vague predicates
$\mathscr{P}$	the space of possible precification points p
$\mathcal{I}$	the interpretation function s.t. for each $R \in \mathcal{R}$:
$\mathcal{I}^+(R,p)$	is the extension of R at p
$\mathcal{I}^-(R,p)$	is the anti-extension of R at p
	$\mathcal{I}^+(R,p)$ and $\mathcal{I}^-(R,p)$ are disjoint
$\leq$	is a partial order with minimal element $\mathscr{M}$ s.t. for all $p, q \in \mathscr{P}, R \in \mathcal{R}$: if $p \leq q$ then $\mathcal{I}^+(R,p) \subseteq \mathcal{I}^+(R,q)$ and $\mathcal{I}^-(R,p) \subseteq \mathcal{I}^-(R,q)$

Delineation approaches

Kyburg/Morreau (2000), Fitting words: Vague language in context

Penumbral connections

$\mathscr{P}$ may be defined by fixing a root element $\mathscr{M}$ and taking all possible precifications constrained by penumbral connections.

Example penumbral connections

Let $\mathscr{R} = \{\mathit{TALL}, \mathit{TALLER\_THAN}\}$. A precification is admissible only if

$$\forall x \forall y. \neg(\mathit{TALLER\_THAN}(x,y) \land \mathit{TALL}(y) \land \neg \mathit{TALL}(x))$$

holds at all points $p \in \mathscr{P}$.

Completability requirement ($\rightarrow$ Supervaluation)

For each partial precification $p \in \mathscr{P}$ there must exist a complete precification $c \in \mathscr{P}$ such that $p \leq c$.

Delineation approaches

Kyburg/Morreau (2000), Fitting words: Vague language in context

- New information may interfere with current one
- e.g. shifting of vague standards

Context update

Assume $p \in \mathscr{P}$ to be the current context. $KB(p)$ is defined as the set of sentences true at p. When confronted with new information s, perform an update of $KB(p)$ as follows:

- Retain all non-vague information in p (all sentences which are true at all $q \in P$),
- add s,
- add as many of the vague sentences of $KB(p)$ as possible without violating a penumbral connection.

- In general, belief revision is ambiguous
- Belief revision theory provides different update operators

Delineation approaches

Concerning vague quantities and quantifiers

- As for Barker's approach, enhance the language by vague quantifiers and quantities:
- "*Many tall people are heavy*" $\rightarrow$ $\mathbf{Many}(\mathbf{tall}, \mathbf{heavy})$
- "Roughly 50 people are tall" $\rightarrow$ $\mathbf{Roughly}(50, \mathbf{tall})$
- At each preciification point $p \in \mathscr{P}$, $\mathbf{Many}(A, B)$ is either true, false, or unsettled

Penumbral connections

- $D(Many(A, B) \rightarrow Many(C, D))$
  if $|A \cap B|/|A| \leq |C \cap D|/|C|$
- $D(\neg Many(A, B) \rightarrow \neg Many(C, D))$
  if $|A \cap B|/|A| \geq |C \cap D|/|C|$
- $D(Many(A, B) \rightarrow \neg Few(A, B))$, $D(Few(A, B) \rightarrow \neg Many(A, B))$

Relating scale- and delineation-based approaches

Overview

Central question

Scale- and delineation-based approaches differ substantially, but precisely in which situations do they give different predictions?

As it turns out:

- under certain conditions (the initial context C_0 and the precification space $\mathscr{P}$ are modeled “using the same assumptions”,...) both approaches make the same predictions
- scale-based approaches allow for more fine-grained predicate modifiers,
- delineation approaches are superior when new information is inconsistent with the current one.

Relating scale- and delineation-based approaches

Translation functions

- Find a common common context representation for both approaches: Sets of classical worlds

Definition

Let $\mathcal{U}$ be the set of individuals and $\mathcal{R}$ the set of vague predicates under consideration. Then a classical world s is a complete interpretation of all the predicates in $\mathcal{R}$ formalized as a set of literals such that for all $R \in \mathcal{R}$ and for all $u \in \mathcal{U}$ either $R(u) \in s$ or $\neg R(u) \in s$.

- Given an initial context C_0 and a context space $\mathscr{P}$ define translation functions from contexts to sets of possible worlds and back for both approaches.

Relating scale- and delineation-based approaches

- For Kyburg/Morreau

Definition

Let $\mathscr{P}$ be a precification space, $p \in \mathscr{P}$ a partial interpretation, and $\mathcal{U}$ the universe of discourse. Then the translation of p to a set of possible worlds, denoted as $T_{km}p$, is defined as

$$T_{km}p =_{\text{DEF}} \bigcirc_{R\in\mathcal{R}} \bigcirc_{u\in\mathcal{U}} \phi(R,u,p) \text{ where}$$

$$\phi(R,u,p) = \begin{cases} \{R(u)\} & \text{iff } u \in \ell^{+}(R,p) \\ \{\neg R(u)\} & \text{iff } u \in \ell^{-}(R,p) \\ \{R(u), \neg R(u)\} & \text{otherwise} \end{cases}$$

for all $R \in \mathcal{R}$.

Relating scale- and delineation-based approaches

Translation functions

- For Kyburg/Morreau

Proposition

Let $\mathscr{P}$ be a precification space and S a set of complete interpretations. Then a partial interpretation $p \in \mathscr{P}$ is identified with S, denoted $p = T_{km}^{-1} S$, if and only if for all $R \in \mathcal{R}$ both

$$\ell^{+}(R, p) = \bigcap_{s \in S} \ell^{+}(R, s) \qquad \text{and} \qquad \ell^{-}(R, p) = \bigcap_{s \in S} \ell^{-}(R, s)$$

hold, thus $T_{km}(T_{km}^{-1} S) = S$.

- NB: If we had defined $T_{km}(p)$ as the set of all complete precisifications $q \in \mathscr{P}$ of p, $T_{km}(T_{km}^{-1} S) = S$ would not hold in general.

Relating scale- and delineation-based approaches

Translation functions

- For Barker

Definition

Let C be a context according to Barker. Then the translation of C to (possible) classical interpretations $T_b C$ is defined as

$$
\begin{aligned}
T_b C &= \{s(w) : w \in C\} \text{ where} \\
R(u) \in s(w) &\text{ iff } w \in [\![R]\!](u)(C) \text{ and} \\
\neg R(u) \in s(w) &\text{ iff } w \notin [\![R]\!](u)(C)
\end{aligned}
$$

for all individuals $u \in \mathcal{U}$ and $R \in \mathcal{R}$.

- analogously for n-ary predicates
- easy, as a context according to Barker already consists of complete precifications.

Relating scale- and delineation-based approaches

Translation functions

- For Barker

Proposition

Let S be a complete interpretation, C_0 a fixed context as defined by Barker. Then a context $C \subseteq C_0$ can be determined such that $T_b C = S$ by setting

$$T_b^{-1} S = \{w \in C_0 : \exists s \in S . \forall u \in \mathcal{U} . \forall R \in \mathcal{R} . w \in [\![R]\!](u)(C_0)\}.$$

- analogously for n-ary predicates
- select those possible worlds in C_0 for which a corresponding classical world in S exists

Relating scale- and delineation-based approaches

Corresponding models

Using composite translation context can be translated from one model into the other:

- For $p \in \mathscr{P}$ one can find $C = T_b^{-1} T_{km} p$ and
- For $C \subseteq C_0$ one can find $p = T_{km}^{-1} T_b C$.

But when exactly are $\mathscr{P}$ and C_0 modeled making the same assumptions?

- Even more importantly:

What about assumptions about scale structure?

Relating scale- and delineation-based approaches

Corresponding models

- In Barker's approach, scale structure is implicitly fixed
- Kyburg and Morreau's approach is initially agnostic of scales
- Solution: determine scale structure by penumbral connections

Linear scale (*tall*)

Characterize the binary (non-vague) predicates **taller_than** and **as_tall_as** and the unary vague one **tall** as:

(NV_1) $D(\forall x \forall y.(\mathbf{as\_tall\_as}(x,y) \vee \neg \mathbf{as\_tall\_as}(x,y))$

(RE_1) $D(\forall x.\mathbf{as\_tall\_as}(x,x))$

(TR_1) $D(\forall x \forall y \forall z.(\mathbf{as\_tall\_as}(x,y) \wedge \mathbf{as\_tall\_as}(y,z)) \to \mathbf{as\_tall\_as}(x,z))$

(SY_1) $D(\forall x \forall y.(\mathbf{as\_tall\_as}(x,y) \to \mathbf{as\_tall\_as}(y,x))$

(NV_2) $D(\forall x \forall y.(\mathbf{taller\_than}(x,y) \vee \neg \mathbf{taller\_than}(x,y))$

(TR_2) $D(\forall x \forall y \forall z.(\mathbf{taller\_than}(x,y) \wedge \mathbf{taller\_than}(y,z))$
$\to \mathbf{taller\_than}(x,z))$

(TI_2) $D(\forall x \forall y.\mathbf{taller\_than}(x,y) \vee \mathbf{taller\_than}(y,x) \vee \mathbf{as\_tall\_as}(x,y))$

Relating scale- and delineation-based approaches

Corresponding models

- In Barker's approach, scale structure is implicitly fixed
- Kyburg and Morreau's approach is initially agnostic of scales
- Solution: determine scale structure by penumbral connections

Linear scale (*tall*) (continued)

Characterize the binary (non-vague) predicates **taller_than** and **as_tall_as** and the unary vague one **tall** as:

$$(i) \quad D(\forall x \forall y.(\mathbf{tall}(x) \wedge \mathbf{taller\_than}(y,x)) \rightarrow \mathbf{tall}(y))$$

$$(ii) \quad D(\forall x \forall y.(\neg\mathbf{tall}(x) \wedge \mathbf{taller\_than}(x,y)) \rightarrow \neg\mathbf{tall}(y))$$

Relating scale- and delineation-based approaches

Corresponding models

Definition

Let C_0 be a context as defined by Barker and $\mathscr{P}$ be a precification space in the sense of Kyburg and Morreau. C_0 and $\mathscr{P}$ are called corresponding contexts if the following conditions are met:

- for each vague predicate p in C_0 there is predicate P in $\mathcal{R}$ and vice versa,
- for each individual **a** in C_0 there is an object a in $\mathcal{U}$ and vice versa,
- for each vague predicate $P \in \mathcal{R}$ the scale of the corresponding predicate p in C_0 is expressed in $\mathscr{P}$ via penumbral connections,
- for each $m \in \mathscr{P}$ there exists $C = T_b^{-1} T_{km} m \subseteq C_0$, and
- for each $C \subseteq C_0$ there exists $m = T_{km}^{-1} T_b C \in \mathscr{P}$.

Proposition

Let C_0 and $\mathscr{P}$ be two corresponding contexts and let s be a proposition of the form "**a** *is p*" such that $[\![p]\!](\mathbf{a})(C_0) \neq \emptyset$.
Then there exists a unique, most general partial interpretation $m \in \mathscr{P}$ such that $P(a)$ is true at m and the translations to complete interpretations $T_b([\![p]\!](\mathbf{a})(C_0))$ and $T_{km}m$ coincide.

- The condition $[\![p]\!](\mathbf{a})(C_0) \neq \emptyset$ is crucial, as both approaches deal differently with inconsistent information
- Analogous results for propositions of the form "**a** *is not p*" and "**a** *is more p than* **b**"

Relating scale- and delineation-based approaches

Equivalence of approaches

Main result

Let C_0 and $\mathscr{P}$ be two corresponding contexts and consider a sequence $s_1, \ldots, s_n$ of propositions of the form **a** *is p*, **a** *is not p*, and **a** *is more p than* **b**. Then, after updating C_0 and m_0 with all of s_1 to s_n resulting in the contexts C and m, respectively, an additional proposition s is true at C if and only if s is true at m.

- Proof follows immediately by noting that after each update the resulting contexts can be translated into the same set of classical interpretations.

Conclusions and further work

We have seen that:

- Although scale- and delineation-based approches differ fundamentally, for simple propositions and consistent information they make the same predictions,
- scale-based approaches allow for more fine-grained predicate modifiers such as *very*, *clearly*, *roughly*,...
- delineation approaches allow for handling of inconsistent information.

Further work comprises:

- Combine both approaches, e.g. for belief revision: Use quantitative data from Barker's contenxts to decide which propositions to hold and which to discard when performing an update with incosnsistent information
- Combine Kyburg/Morreau's approach with Shapiro's work on vagueness in context instead of supervaluation; allows for formulating stronger penumbral connections, e.g. for the Sorites paradox.

That's it

Thanks for your attention!

Any questions?